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LETTER TO THE EDITOR

Bi-Hamiltonian structure and Lie–Bäcklund symmetries for a modified Harry–Dym system

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Abstract. We have obtained the complete Lie-Bäcklund symmetry for a modified Harry-Dym system and hence deduced the bi-Hamiltonian structure associated with it. It is shown that these Lie-Bäcklund symmetries generate the recursion operator inherent in the theory and the conserved quantities can also be computed according to the Dorfman prescription.

One of the most important properties of nonlinear systems which are completely integrable is possession of an infinite number of conservation laws [1]. Another notable feature associated with such equations is their bi-Hamiltonian structure, connected intimately with the recursion operator for these infinite number of symmetries [2,3]. In this letter we analyse such properties in relation to an equation which can be thought of as the modified Harry-Dym system. The equation under consideration reads

$$U_t = \partial x^3 (C_1 + C_2 U)^{-1/2} - (D_1 + D_2 U)(C_1 + C_2 U)^{-3/2} U_x + \tfrac{1}{2} D_2 C_2 (C_1 + C_2 U)^{-1/2} U_x \tag{1}$$

which for brevity we put as

$$U_t = K(U, U_x, U_{xx}, \ldots). \tag{2}$$

The Lie-Bäcklund symmetries associated with either of the forms (1) or (2) are generated through a generator $\eta(U, U_x, U_{xx} \ldots)$ such that when

$$U \to U + \varepsilon\eta$$

equation (2) remains invariant. Mathematically, a functional can be a symmetry of (2) if and only if

$$K'[\eta] - \eta'[K] = 0 \tag{3}$$

where $K'[\nu]$ is defined to be the gradient of K, through the equation

$$K'[\nu] = \langle \text{grad } K, \nu \rangle \tag{4}$$

with

$$K'[\nu] = (\partial/\partial\varepsilon) K[U + \varepsilon\nu]\big|_{\varepsilon=0}$$

and $\langle\, ,\rangle$ is the relevant scalar product.

In our case of the modified Harry-Dym system the function K is written as

$$K(U)=\partial x^3(C_1+C_2U)^{-1/2}-(D_1+D_2U)(C_1+C_2U)^{-3/2}U_x$$
$$+\tfrac{1}{2}D_2C_2(C_1+C_2U)^{-1/2}U_x \tag{5}$$

and

$$K'[\nu]=-\tfrac{1}{2}C_2\,\partial x^3(C_1+C_2U)^{-3/2}+\tfrac{1}{2}D_2C_2(C_1+C_2U)^{-1/2}$$
$$+(D_1+D_2U)(C_1+C_2U)^{-3/2}. \tag{6}$$

With this form of $K'[\nu]$ it can be seen that the first few solutions of (3) are

$$\eta_1=U_x$$
$$\eta_3=K(U)$$
$$\eta_5=LQ_1$$

with $Q_1=U_2f(U,U_x)+g(U,U_x)$, where

$$L=\partial_x^3+2(D_1+D_2U)\,\partial_x+D_2U_x$$
$$f=-\tfrac{1}{8}C_2^2(C_1+C_2U)^{-5/2}$$
$$g=\tfrac{5}{32}C_2^3(C_1+C_2U)^{-7/2}U_1^2+\tfrac{1}{4}C_2(D_1+D_2U)(C_1+C_2U)^{-3/2}$$

and lastly

$$\eta_7=LQ_2$$

where

$$Q_2=U_4h+U_2^2k+lU_2+m$$

with

$$h=-\tfrac{1}{16}C_2^2(C_1+C_2U)^{-7/2}$$
$$l=-\tfrac{35}{64}C_2^4(C_1+C_2U)^{-11/2}U_1^2+\tfrac{1}{8}D_2C_2(C_1+C_2U)^{-5/2}-\tfrac{5}{16}(C_1+C_2U)^{-7/2}(D_1+D_2U)$$
$$k=\tfrac{35}{64}C_2^3(C_1+C_2U)^{-9/2}$$
$$m=\tfrac{1155}{752}C_2^5(C_1+C_2U)^{-13/2}U_1^4-\tfrac{15}{16}D_2C_2^2(C_1+C_2U)^{-7/2}U_1^2$$
$$+\tfrac{3}{16}C_2(D_1+D_2U)^2(C_1+C_2U)^{-5/2}$$
$$+\tfrac{35}{64}C_2^3(C_1+C_2U)^{-9/2}(D_1+D_2U)U_1^2. \tag{7}$$

The structure of these symmetries indicates that the further symmetries will be of quite complicated structures. One of the most useful properties of these types of equations, possessing infinite number of symmetries, is that they admit an hereditary operator [4,1]. Let us recapitulate the definition of such an operator, Φ. An operator Φ is said to be a strong symmetry or an hereditary operator if

$$\Phi[K]-[K',\Phi]=0 \tag{8}$$

and an operator Φ satisfying (8) has the property

$$\eta_i=\Phi\eta_{i-1}. \tag{9}$$

In this case we find that Φ has the following form:

$$\begin{aligned}\Phi = {} & \tfrac{15}{16}(\alpha^{-1/2}D^{-1}\alpha^{-7/2}C_2^3U_1^3-\alpha^{-7/2}C_2^3U_1^3D^{-1}\alpha^{-1/2})\\ & +\tfrac{9}{8}(\alpha^{-5/2}C_2^2U_1U_2D^{-1}\alpha^{-1/2}-\alpha^{-1/2}D^{-1}\alpha^{-5/2}C_2^2U_1U_2)\\ & +\tfrac{9}{16}(\alpha^{-5/2}C_2^2U_1^2D^{-1}\alpha^{-3/2}C_2U_1-\alpha^{-3/2}C_2U_1D^{-1}\alpha^{-5/2}C_2^2U_1^2)\\ & +\tfrac{3}{8}[\alpha^{-3/2}(C_2U_1D^{-1}\alpha^{-3/2}C_2U_2-C_2U_2D^{-1}\alpha^{-3/2}C_2U_1)]\\ & +\tfrac{1}{4}(\alpha^{-1/2}D^{-1}\alpha^{-3/2}C_2U_3-\alpha^{-2}C_2U_3)+2\alpha^{-1}\partial_x^2\\ & +\tfrac{11}{7}\alpha^{-2}C_2U_1\partial_x+3\alpha^{-3}C_2^2U_1^2-2\alpha^{-2}C_2U_2\\ & -\tfrac{1}{2}(D_1+D_2U)\alpha^{-3/2}C_2U_1D^{-1}\alpha^{-1/2}+2(D_1+D_2U)\alpha^{-1}\\ & +\tfrac{1}{2}(D_1+D_2U)\alpha^{-1/2}D^{-1}\alpha^{-3/2}C_2U_1+\tfrac{1}{2}D_2U_1\alpha^{-1/2}D^{-1}\alpha^{-1/2}\end{aligned}\qquad(10)$$

where $\alpha = C_1 + C_2U$. This satisfies both equations (8) and (9).

An interesting and crucial observation at this stage is that (10) factorises in the following way:

$$\Phi = [\partial_x^3 + 2(D_1 + D_2U)\,\partial_x + D_2U_x]M^{-1} = LM^{-1} \qquad (11)$$

with

$$M = 2(C_1 + C_2U)\,\partial x + C_2U_x. \qquad (12)$$

It is then easy to demonstrate that the Q_i defined in equation (7) follow the recursion relation

$$MQ_{i+1} = LQ_i. \qquad (13)$$

Equation (13) is really the clue to the bi-Hamiltonian structure. Let us define quantities $\mathcal{G}_i$ whose variational derivatives are the quantities Q_i, that is

$$\delta\mathcal{G}_i/\delta U = Q_i \qquad (14)$$

where

$$\frac{\delta}{\delta U} = \sum_0^\infty (-\partial_0)^i \frac{\partial}{\partial U_i} \qquad \partial_0 = \sum_0^\infty U_{i+1}\frac{\partial}{\partial U_i}$$

are really the conserved quantities associated with the symmetries η_i. In fact equations of the form (14) can be solved for $\mathcal{G}_i$ in the variational formalism of Dorfman [5]. The solution of equation (14) can be actually written as

$$\mathcal{G}_i = U\int_0^1 Q_i(tU)\,\mathrm{d}t. \qquad (15)$$

The construction of a bi-Hamiltonian structure can then be realised by specifying two sets of Poisson brackets along with two Hamiltonians as follows. Let us set for any two functionals F_i, F_j of U

$$\begin{aligned}[F_i, F_j]_1 &= \int \frac{\delta F_i}{\delta U} L \frac{\delta F_j}{\delta U}\,\mathrm{d}x\\ [F_i, F_j]_2 &= \int \frac{\delta F_i}{\delta U} M \frac{\delta F_j}{\delta U}\,\mathrm{d}x\end{aligned}\qquad (16)$$

where L and M are the two operators in (7) and (12). Then equation (1) can be seen to be equivalent to

$$U_t = [U, H_1]_1 \equiv [U, H_2]_2 \tag{17}$$

where $H_1 = \mathscr{G}_0 = (C_1 + C_2 U)^{-1/2}$ and $H_2 = \mathscr{G}_1$ obtained from (15) to be equal to

$$-\tfrac{1}{12} C_2 (C_1 + C_2 U)^{-3/2} U_2 - \tfrac{1}{16} C_2^2 (C_1 + C_2 U)^{-5/2} U_1^2$$
$$-\tfrac{1}{2}(D_1 + D_2 U)(C_1 + C_2 U)^{-1/2} + (D_2/C_2)(C_1 + C_2 U)^{1/2}.$$

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